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LETTER TO THE EDITOR

Painlevé analysis and particular solutions of a coupled nonlinear reaction diffusion system

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Abstract. We consider the following nonlinear coupled partial differential equations which describe a biochemical model system;

$$\partial x/\partial t = x^2 y - Bx + K\partial^2 x/\partial r^2$$

$$\partial y/\partial t = -x^2 y + Bx + k\partial^2 y/\partial r^2$$

where B is a parameter and K is the diffusion constant for the concentrations x and y. Seeking a wave solution, we obtain a nonlinear ordinary differential equation in one dynamical variable only. This ODE is analysed for the Painlevé property and then we are able to find a one-parameter family of solutions which are of biological interest.

It is a strong belief nowadays that the Painlevé property of nonlinear differential equations (the singularities of whose solutions are movable simple poles only) is closely connected with the integrability or solvability of the equations. It is noted that all evolution equations possessing soliton solutions are reduced to ordinary differential equations (ODEs) by the similarity variables and these ODEs have the Painlevé property (Ablowitz *et al* 1978, 1980, Lakshmanan and Kaliappan 1983). It is quite interesting to search for the Painlevé property for those equations for which soliton solutions have not, so far, been found and try to find some explicit solutions. In this letter, we consider one such pair of nonlinear coupled partial differential equations which describe a biochemical model system (Lefever *et al* 1977).

$$\partial x/\partial t = x^2 y - Bx + K\partial^2 x/\partial r^2 \tag{1a}$$

$$\partial y/\partial t = -x^2 y + Bx + K\partial^2 y/\partial r^2 \tag{1b}$$

where B is a parameter and K is the diffusing constant for the concentrations x and y. Seeking a wave solution, we obtain a nonlinear ODE, by reducing the system to an equation in one dynamical variable only. This ODE is analysed for the Painlevé property and then we are able to find a one-parameter family of solutions which are of biological interest.

Adding (1a) and (1b) we get

$$\frac{\partial(x+y)}{\partial t} = K\frac{\partial^2(x+y)}{\partial r^2}. \tag{2}$$

§ Deceased.

A simple solution to (2) can be taken as

$$x+y=K_1 \qquad \text{(a constant).} \tag{3}$$

Putting this in ($1a$) we get an equation in the variable x only as

$$\partial x/\partial t = K\partial^2 x/\partial r^2 - Bx + K_1 x^2 - x^3. \tag{4}$$

On seeking a wave solution, we let $z = r + \lambda t$. This yields the following ODE (by adjusting the constants)

$$\frac{d^2x}{dz^2} - \lambda \frac{dx}{dz} - ax + bx^2 - x^3 = 0 \tag{5}$$

where a, b are constants. Incidentally if we take $b = a+1$, we obtain a Nagumo-type equation (Mckean 1970)

As in the procedure given by Ablowitz and Zepetella (1979), and Airault and Kaliappan (1984) we shall find the values of λ for which equation (5) has a solution developable in Laurent series of the form $a_{-n}z^{-n} + a_{-n+1}z^{-n+1} + \ldots$. In this case $n=1$ and $(a_{-1})^2 = 2$. Let us search for the solutions of the form

$$x = z^{-1}\sqrt{2} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots.$$

Putting this in (5) and equating the corresponding coefficients of z, we obtain the following conditions:

$$\lambda = \sqrt{2}\,(3a_0 - b) \tag{6}$$

$$6a_1 = -a\sqrt{2} + 2\sqrt{2}\,ba_0 - 3\sqrt{2}\,a_0^2 \tag{7a}$$

$$-4a_2 = \lambda a_1 + aa_0 - a_0^2 b - 2\sqrt{2}\,a_1 b + a_0^3 + b\sqrt{2}\,a_1 a_0 \tag{7b}$$

$$0a_3 - 2\lambda a_2 - aa_1 + 2a_0 a_1 b + 2\sqrt{2}\,a_2 b - 3\sqrt{2}\,a_1^2 - 3a_0^2 a_1 - b\sqrt{2}\,a_0 a_2 = 0. \tag{8}$$

The compatibility condition will be satisfied if we have the following three values of λ:

$$\lambda = b/\sqrt{2}$$

$$\lambda = \{-b + 3\sqrt{b^2 - 4a}\}/2\sqrt{2}$$

$$\lambda = \{-b - 3\sqrt{b^2 - 4a}\}/2\sqrt{2}.$$

Thus we find equation (5) to possess the Painlevé property for the above values of λ.

Since the Painlevé property is a sufficient condition for the integrability of a nonlinear partial differential equation, we conclude that the equation (5) is integrable and has explicit solutions, (Wiess 1983, Kaliappan 1984). We proceed below to find the same, by defining a subequation. However, it must be stressed here that the Painlevé property is only a sufficient condition, but not a necessary condition, for a nonlinear partial differential equation to have explicit solutions (Wiess *et al* 1983).

Particular solutions: The second-order equation (5) has a sub-equation of order 1: $dx/dz = Ax^2 + Bx + C$. We get $2A^2 = 1$ and we find that the only possible subequations are

$$\frac{dx}{dz} = Ax\left\{x - \frac{b + \sqrt{b^2 - 4a}}{2}\right\} \qquad \text{with } \lambda = \frac{-A(-b + 3\sqrt{b^2 - 4a})}{2} \tag{10a}$$

$$\frac{dx}{dz}=Ax\left\{x-\frac{b-\sqrt{b^2-4a}}{2}\right\} \qquad \text{with } \lambda=\frac{-A(-b-3\sqrt{b^2-4a})}{2} \tag{10b}$$

$$\frac{dx}{dz}=A\left\{x-\frac{b+\sqrt{b^2-4a}}{2}\right\}\left\{x-\frac{b-\sqrt{b^2-4a}}{2}\right\} \qquad \text{with } \lambda=-bA. \tag{10c}$$

Taking $A=-1/\sqrt{2}$ the solutions for (10a), (10b) and (10c) are found as follows:

$$x=L\alpha\exp(\alpha z/\sqrt{2})/[\exp(\alpha z/\sqrt{2})-1] \qquad \alpha=[b+\sqrt{b^2-4a}]/2 \tag{11a}$$

$$x=L\beta\exp(\beta z/\sqrt{2})/[\exp(\beta z/\sqrt{2})-1] \qquad \beta=[b-\sqrt{b^2-4a}]/2 \tag{11b}$$

$$x=\{\alpha-\beta L\exp[-\sqrt{b^2-4a}\,Z/\sqrt{2}]\}/\{1-L\exp[-\sqrt{b^2-4a}\,z/\sqrt{2}]\} \tag{11c}$$

where L is the integration constant.

When we take $b=a+1$, these solutions are reduced to the solutions found by Airault and Kaliappan (1984). Further by taking $b=3/\sqrt{2}$, $a=1$ and by the transformations $x=(x'+1)/\sqrt{2}$ and $z'=z/\sqrt{2}$ equation (5) reduces to (after dropping primes)

$$\frac{d^2x}{dz^2}-\lambda\sqrt{2}\frac{dx}{dz}+x-x^3=0 \tag{12}$$

which is of Fisher-type equation with Cubic nonlinearity.

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